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A symmetric generalization of linear Bäcklund transformation associated with the Hirota bilinear difference equation

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Abstract. The Hirota bilinear difference equation is generalized to discrete space of arbitrary dimension. Solutions to the nonlinear difference equations can be obtained via Bäcklund transformation of the corresponding linear problems.

1. Introduction

The Hirota bilinear difference equation (HBDE) [1] plays a central role in the study of integrable nonlinear systems. This single equation embodies an infinite number of integrable differential equations which belong to the KP-hierarchy as shown by Miwa [2]. It characterizes algebraic curves [3], appears as a consistency relation for Laplace maps on a discrete surface [4] and is satisfied by string correlation functions in particle physics [5]. It is also satisfied by transfer matrices [6–8], or by certain quantities properly defined from Boltzmann weights [9], of some solvable lattice models.

Solutions to the HBDE, called τ functions, are known [10, 11] to form a Grassmann manifold of infinite dimensions. On this space of solutions a Bäcklund transformation acts and generates the $GL(\infty)$ symmetry. This large symmetry is the origin of integrability of the system. The scheme of the Bäcklund transformation can be seen most easily if we convert HBDE into a pair of linear homogeneous equations. It was first found by Hirota in [1] and then represented in a manifestly symmetric form under the Bäcklund transformations in [12]. The coupled linear equations describe the behaviour of a wavefunction under the influence of gauge potentials. The compatibility condition for the wavefunction to solve the coupled equations requires the gauge potential to satisfy the HBDE.

The situation is similar to the inverse scattering method. The difference from the ordinary inverse method is that every solution to the linear equations also fulfils the same nonlinear equation satisfied by the gauge potential. This is due to the remarkable symmetry under the exchange of the wavefunction and the gauge potential, which we called the dual symmetry in [12]. Hence, by solving the linear system of equations iteratively we obtain a series of solutions to the nonlinear equation. We call this scheme of finding new solutions to a nonlinear equation, a linear Bäcklund transformation (LBT).

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An interesting application of this scheme was found and described in [8]. The authors showed that it can be used to generate Bethe ansatz solutions to certain classes of solvable lattice models. In order to uncover the meaning of the correlation between the solvable lattice models and the HBDE, it is desirable to formulate a LBT such that the symmetries of this scheme can be seen from both sides. From this point of view we discuss, in this paper, a symmetric generalization of LBTs.

The HBDE is a highly symmetric equation under the exchange of lattice variables. The linear equations discussed in [12], however, do not possess this symmetry. In section 3 we derive a symmetric version of the LBT associated with HBDE. We will show in section 4 that we can generalize this scheme to a lattice space of arbitrary dimension. Corresponding to this generalization we obtain a large number of nonlinear equations which can be solved by the LBT method. The HBDE turns out to be a special case of this scheme.

2. Hirota bilinear difference equation

Let us begin by writing the Hirota bilinear difference equation (HBDE) [1] for a function $f \in C^\infty$ of discrete variables λ, μ, ν :

$$\alpha f(\lambda + 1, \mu, \nu) f(\lambda - 1, \mu, \nu) + \beta f(\lambda, \mu + 1, \nu) f(\lambda, \mu - 1, \nu) + \gamma f(\lambda, \mu, \nu + 1) f(\lambda, \mu, \nu - 1) = 0 \quad (1)$$

where α, β, γ are arbitrary complex parameters. This is a very simple and highly symmetric equation. In order to show how the LBT generates solutions to (1), we follow the argument of [5], but in a different notation convenient later.

We consider the following coupled linear problems:

$$\nabla_{12} g = \omega_{12} g \quad \nabla_{21} g = \omega_{21} g \quad (2)$$

$$\nabla_{12} := f_2 e^{\partial_\lambda - \partial_\nu} f_2^{-1} - c_1 f_1 e^{\partial_\mu - \partial_\nu} f_1^{-1} \quad (3)$$

$$\nabla_{21} := f_1 e^{-\partial_\lambda - \partial_\nu} f_1^{-1} - c_2 f_2 e^{-\partial_\mu - \partial_\nu} f_2^{-1}$$

where f_1, f_2 are functions of λ, μ, ν and $\omega_{12}, \omega_{21}, c_1, c_2$ are constants. The shift operator e^{∂_x} acts to all functions on the right by changing x to $x + 1$.

HBDE (1) emerges from the compatibility condition of this set of linear equations

$$[\nabla_{12}, \nabla_{21}] = 0. \quad (4)$$

In fact, if we impose the condition $f_1(\lambda, \mu + 1, \nu + 1) = f_2(\lambda + 1, \mu, \nu + 1) =: f(\lambda, \mu, \nu)$ to the gauge potentials, this operator relation turns to

$$e^{-2\partial_\nu} \frac{f(\lambda, \mu - 1, \nu + 1) f(\lambda - 1, \mu, \nu + 1)}{f(\lambda - 1, \mu - 1, \nu) f(\lambda, \mu, \nu)} (e^{-\partial_\mu} - e^{-\partial_\lambda}) \times \left\{ \frac{f(\lambda + 1, \mu, \nu) f(\lambda - 1, \mu, \nu)}{f(\lambda, \mu, \nu + 1) f(\lambda, \mu, \nu - 1)} - c_1 c_2 \frac{f(\lambda, \mu + 1, \nu) f(\lambda, \mu - 1, \nu)}{f(\lambda, \mu, \nu + 1) f(\lambda, \mu, \nu - 1)} \right\} = 0. \quad (5)$$

Since the left-hand side is a difference of the quantity in the brace, this quantity itself must be a constant. By adjusting the parameters as $c_1 c_2 = -\beta/\alpha$, and putting the constant $-\gamma/\alpha$ we obtain the HBDE.

An important observation is that the linear equations (2) are symmetric under the exchange of the role of the gauge potentials f and the wavefunction g . This is due to the fact that under the same constraints for the gauge fields f 's, (2) can also be written as

$$\tilde{\nabla}_{12} f = \omega_{12} f \quad \tilde{\nabla}_{21} f = \omega_{21} f \quad (6)$$

if we define $g_1(\lambda, \mu - 1, \nu - 1) = g_2(\lambda - 1, \mu, \nu - 1) = g(\lambda, \mu, \nu)$ and

$$\begin{aligned} \tilde{V}_{12} &:= g_2 e^{-\partial_\lambda + \partial_\nu} g_2^{-1} - c_1 g_1 e^{-\partial_\mu + \partial_\nu} g_1^{-1} \\ \tilde{V}_{21} &:= g_1 e^{\partial_\lambda + \partial_\nu} g_1^{-1} - c_2 g_2 e^{\partial_\mu + \partial_\nu} g_2^{-1}. \end{aligned} \tag{7}$$

It is not difficult to convince ourselves that the linear equations (6) for the new wavefunction f can be solved only if the new gauge potential g satisfies (1), the same equation satisfied by f . This is what we called dual symmetry in [12].

The LBT works as follows. Starting from a particular solution to HBDE, say $f^{(1)}$, as a gauge potential we solve the linear problem (2). One of its solutions, which we call $g^{(1)}$ satisfies the HBDE because it can be regarded as a potential of the coupled linear equations (6) which are nothing but another expression of (2).

The set of coupled equations (6) should have solutions other than $f^{(1)}$. Let us call it $f^{(2)}$. $f^{(2)}$ must also satisfy HBDE, because (2) has $g^{(1)}$ as a solution, hence it is compatible. Now we can repeat the same argument starting from $f^{(2)}$ to obtain a new solution $g^{(2)}$ to the HBDE, and so on. This is an auto-Bäcklund transformation since f 's and g 's are solutions to the same equation. By starting from the simplest solution $f^{(1)} = 1$ in the case of $\alpha + \beta + \gamma = 0$ in (1), for instance, we obtain a series of soliton solutions explicitly in this method [12].

3. Symmetrization of the LBT

The HBDE is highly symmetric by itself, in contrast to its continuous reductions, such as the KdV equation, Toda lattice equation, sine-Gordon equation etc. The corresponding linear version of HBDE, (2) and (6), however, are not symmetric. In this section, we derive a complete set of linear equations which recover the symmetries possessed by the HBDE.

First we examine the symmetries of the HBDE. From (1) HBDE relates functions defined on the six lattice sites $(\lambda \pm 1, \mu, \nu)$, $(\lambda, \mu \pm 1, \nu)$, $(\lambda, \mu, \nu \pm 1)$, which form an octahedron in the three-dimensional lattice space. Connecting these corners of octahedrons forms a face centred cubic (fcc) lattice. Hence dependent variables of the HBDE reside on a fcc lattice, rather than a simple cubic lattice. Indeed the vectors \mathbf{p} , \mathbf{q} , \mathbf{r} in figure 1 are the primary translation vectors of a fcc.

The HBDE possesses point group symmetries which transform an octahedron to itself. It includes the inversion I , which changes the sign of the variables λ, μ, ν , and the mirror reflections σ_{ij} which exchange the triangles S_i and S_j in figure 2. The LBT represented in the form of (2), however, is not symmetric under the group transformation. The purpose of this section is to make it symmetric.

First note that the potential field f_1 appears in (2) as a gauge field in the form of covariant shift operators $f_1 e^{\partial_\mu - \partial_\nu} f_1^{-1}$ and $f_1 e^{-\partial_\lambda - \partial_\nu} f_1^{-1}$ defined on two edges of the triangle S_1 in figure 2. Similarly the field f_2 is associated with the triangle S_2 . On the other hand from (6) we learn that the fields g_1 and g_2 are associated with the triangles \tilde{S}_1 and \tilde{S}_2 , respectively, which are obtained from S_1 and S_2 via inversions with respect to the origin. Hence equations (2) and (6) are related to each other by the inversion I . In other words the inversion is realized by the dual symmetry. The rest of the point group symmetry requires potential fields associated with S_3 and S_4 to exist. Let us call them f_3 and f_4 , respectively.

In order to make the symmetry manifest it is convenient to introduce the following set

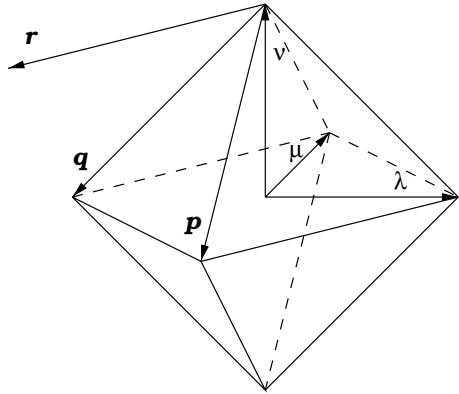


Figure 1.

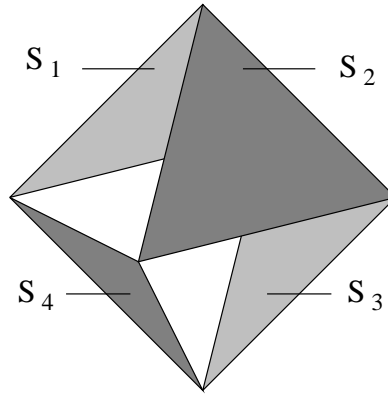


Figure 2.

of derivatives

$$\begin{aligned} \hat{\partial}_1 &= \frac{\partial_\lambda - \partial_\mu - \partial_\nu}{2} & \hat{\partial}_2 &= \frac{\partial_\mu - \partial_\nu - \partial_\lambda}{2} \\ \hat{\partial}_3 &= \frac{\partial_\nu - \partial_\lambda - \partial_\mu}{2} & \hat{\partial}_4 &= \frac{\partial_\lambda + \partial_\mu + \partial_\nu}{2}. \end{aligned} \tag{8}$$

They represent the gradients along the normal to the surfaces S_1, S_2, S_3, S_4 . In terms of these operators the point symmetry transformations can be expressed as

$$\begin{aligned} I : f_j &\leftrightarrow g_j & \hat{\partial}_j &\rightarrow -\hat{\partial}_j \\ \sigma_{ij} : f_i &\leftrightarrow f_j & \hat{\partial}_i &\leftrightarrow \hat{\partial}_j. \end{aligned} \tag{9}$$

Applying σ 's to (2) successively we obtain, up to coefficients, the following six pairs of equations

$$\begin{aligned} (f_j e^{\hat{\partial}_k - \hat{\partial}_i} f_j^{-1} + f_k e^{\hat{\partial}_j - \hat{\partial}_i} f_k^{-1} + 1)g &= 0 \\ (f_k e^{\hat{\partial}_j - \hat{\partial}_i} f_k^{-1} + f_j e^{\hat{\partial}_k - \hat{\partial}_i} f_j^{-1} + 1)g &= 0 \\ j < k & \quad (i, j, k, l) = \text{even permutation of } (1, 2, 3, 4). \end{aligned} \tag{10}$$

Among the 12 equations of (10) note that, corresponding to each triangle of the octahedron, there are three equations which connect two edges of the same triangle by the gauge fields. For instance the set of equations $(j, k) = (1, 2), (2, 4), (1, 4)$ determine the triangle S_3 . For the consistency of these equations they must be the same. We can resolve this problem if we adopt the following conditions to the gauge fields:

$$f_j(\lambda, \mu, \nu) = e^{\hat{\partial}_j - \hat{\partial}_4} f(\lambda, \mu, \nu) \quad j = 1, 2, 3, 4. \tag{11}$$

Using the fact that $\hat{\partial}_1 + \hat{\partial}_2 + \hat{\partial}_3 + \hat{\partial}_4 = 0$ we can write (10) as

$$(e^{-\hat{\partial}_j - \hat{\partial}_i} f)(e^{\hat{\partial}_j + \hat{\partial}_4} g) + (e^{-\hat{\partial}_k - \hat{\partial}_i} f)(e^{\hat{\partial}_k + \hat{\partial}_4} g) + (e^{-\hat{\partial}_i - \hat{\partial}_i} f)(e^{\hat{\partial}_i + \hat{\partial}_4} g) = 0 \tag{12}$$

for all even permutations of (1,2,3,4). Here we used the notation $(e^{\hat{\partial}} f)$ to mean that $e^{\hat{\partial}}$ acts only on the functions in the bracket. In this expression it is obvious that, for a given i , three equations with different choices of j, k, l are the same equation. Therefore we have only four different equations, which we can write as

$$\sum_{j \neq i} a_{ij} (e^{-\hat{\partial}_i - \hat{\partial}_j} f)(e^{\hat{\partial}_j + \hat{\partial}_4} g) = 0 \quad i = 1, 2, 3, 4. \tag{13}$$

We have recovered coefficients a_{ij} in this expression. They are free unless we specify the values of α, β, γ in HBDE. This is the symmetrized LBT which we were looking for and is symmetric under the σ transformations.

The symmetric LBT (13) possesses the σ symmetry as we required. From the analogy of the correspondence between (2) and (6), the inversion symmetry, which has not been imposed so far, should be included if the dual symmetry under the exchange of f and g holds. In fact if we multiply the shift operator $e^{\hat{\partial}_i - \hat{\partial}_4}$ from the left to the i th equation of (13), it becomes

$$\sum_{j \neq i} a_{ij} (e^{\hat{\partial}_i + \hat{\partial}_j} g) (e^{-\hat{\partial}_j - \hat{\partial}_4} f) = 0 \quad i = 1, 2, 3, 4. \tag{14}$$

These are exactly the equations we obtain from (13) by the inversion transformation I of (9). Therefore (13) itself is the totally symmetric LBT which we expected.

Now we will show how HBDE arises from this set of equations. Writing (13) explicitly we have, up to the coefficients,

$$\begin{pmatrix} 0 & f(\lambda, \mu, \nu + 1) & f(\lambda, \mu + 1, \nu) & f(\lambda - 1, \mu, \nu) \\ f(\lambda, \mu, \nu + 1) & 0 & f(\lambda + 1, \mu, \nu) & f(\lambda, \mu - 1, \nu) \\ f(\lambda, \mu + 1, \nu) & f(\lambda + 1, \mu, \nu) & 0 & f(\lambda, \mu, \nu - 1) \\ f(\lambda - 1, \mu, \nu) & f(\lambda, \mu - 1, \nu) & f(\lambda, \mu, \nu - 1) & 0 \end{pmatrix} \times \begin{pmatrix} g(\lambda + 1, \mu, \nu) \\ g(\lambda, \mu + 1, \nu) \\ g(\lambda, \mu, \nu + 1) \\ g(\lambda + 1, \mu + 1, \nu + 1) \end{pmatrix} = 0. \tag{15}$$

Since this is a homogeneous linear equation the determinant of the coefficient matrix must vanish. For simplicity we assume that the matrix is antisymmetric, that is, the coefficients satisfy $a_{ij} = -a_{ji}$. In this case the determinant is the square of Pfaffian, hence the solvability condition turns out to be

$$a_{14}a_{23}f(\lambda + 1, \mu, \nu)f(\lambda - 1, \mu, \nu) - a_{13}a_{24}f(\lambda, \mu + 1, \nu)f(\lambda, \mu - 1, \nu) + a_{12}a_{34}f(\lambda, \mu, \nu + 1)f(\lambda, \mu, \nu - 1) = 0. \tag{16}$$

In this way the HBDE is reproduced.

4. Generalization of HBDE to higher dimensions

Our symmetric LBT (13) possesses the symmetries of the three-dimensional fcc lattice space. The fourth derivative $\hat{\partial}_4$, however, plays a different role in (13) from others. This is due to the fact that there are only three independent variables needed to characterize the four surfaces $S_1 \sim S_4$. This also prevents us from extending the space to higher dimensions.

In this section, we show that the symmetric LBT (13) can be rewritten in a manifestly symmetric form in the four-dimensional lattice space if we introduce a parameter s which specifies the order of Bäcklund transformations. Moreover, it also enables us to generalize the linear equations to arbitrary dimensional lattice space.

Instead of considering two different kinds of fields we associate the parameter $s + 2$ to f and $s + 1$ to g and define a new function $f(\lambda, \mu, \nu; s)$ by

$$f(\lambda, \mu, \nu; s + 2) = f(\lambda, \mu, \nu) \quad f(\lambda, \mu, \nu; s + 1) = g(\lambda, \mu, \nu). \tag{17}$$

We also introduce new variables k_j 's with $j = 1, 2, 3, 4$ by

$$\begin{aligned} k_1 &:= \frac{\lambda - \mu - \nu}{2} + \frac{s}{4} & k_2 &:= \frac{\mu - \nu - \lambda}{2} + \frac{s}{4} \\ k_3 &:= \frac{\nu - \lambda - \mu}{2} + \frac{s}{4} & k_4 &:= \frac{\lambda + \mu + \nu}{2} + \frac{s}{4}. \end{aligned} \quad (18)$$

It is not difficult to convince ourselves that the operations of $\hat{\partial}_j$ is equivalent to the operation of $\partial_j - \partial_s$, where ∂_j means $\partial/\partial k_j$ for all j . Hence the symmetric LBT (13) can be written as

$$\sum_{j=1}^4 a_{ij} (e^{\sum_{l=1}^4 \partial_l - \partial_i - \partial_j} f) (e^{\partial_j} f) = 0 \quad i = 1, 2, 3, 4. \quad (19)$$

Note that (19) is totally symmetric for all variables k_j , $j = 1, 2, 3, 4$.

In terms of the new variables the HBDE turns out to be

$$\begin{aligned} \alpha f(k_1 + 1, k_2, k_3, k_4 + 1) f(k_1, k_2 + 1, k_3 + 1, k_4) \\ + \beta f(k_1, k_2 + 1, k_3, k_4 + 1) f(k_1 + 1, k_2, k_3 + 1, k_4) \\ + \gamma f(k_1, k_2, k_3 + 1, k_4 + 1) f(k_1 + 1, k_2 + 1, k_3, k_4) = 0. \end{aligned} \quad (20)$$

The introduction of the fourth variable does not affect the contents of the equations. However, it enables us to generalize the equations to higher dimensional lattice space. In the following we show that the generalized equations also provide a Bäcklund transformation scheme which generates solutions to new nonlinear difference equations.

For convenience we define

$$f_j = e^{\partial_j} f \quad \tilde{f}_j = e^{-\partial_j} f \quad (21)$$

$$G_{ij}(f) = a_{ij} \exp \left[\sum_{l=1}^n \partial_l - \partial_i - \partial_j \right] f \quad \tilde{G}_{ij}(f) = a_{ij} \exp \left[- \sum_{l=1}^n \partial_l + \partial_i + \partial_j \right] f. \quad (22)$$

Then the n -dimensional LBT and its dual are

$$\sum_{j=1}^n G_{ij}(f) f_j = 0 \quad i = 1, 2, 3, \dots, n \quad (23)$$

$$\sum_{j=1}^n \tilde{G}_{ij}(f) \tilde{f}_j = 0 \quad i = 1, 2, 3, \dots, n. \quad (24)$$

They are the same equation but written differently. The Bäcklund transformation of the system (23) and (24) proceeds in a way similar to the LBT of HBDE, which was discussed in section 2. For the wavefunction f_j to be a solution of the homogeneous equation (23), the potential f must satisfy

$$\det[G(f)] = 0. \quad (25)$$

This is a nonlinear difference equation of f . At the same time the solvability condition of (24) requires

$$\det[\tilde{G}(f)] = 0. \quad (26)$$

Equations (25) and (26) are the nonlinear equations to be solved and to generalize (4). Since (25) and (26) are not the same equation in general, the transformation may not be auto-Bäcklund transformation. In the case of $n = 4$, these two equations are identical and coincide with the square of HBDE, as we have seen above. Hence (25) and (26) are generalization of HBDE to an n -dimensional lattice. Note that at every step of the Bäcklund

transformation the value of the parameter s is changed by one. Since s specifies a hyperplane in the n -dimensional lattice space, the Bäcklund transformation propagates fields on one plane to the next.

Before closing this paper let us make a brief remark on the connection of these results with other works. A generalization of HBDE to a higher-dimensional lattice space has been discussed in [13]. In their paper the equation itself is bilinear and is satisfied by the same τ functions of HBDE. It is a natural extension because the space of solutions to the KP-hierarchy itself is symmetric under the choice of three variables out of infinite number of variables k_1, k_2, k_3, \dots . We have checked that all soliton type of solutions to the higher-dimensional HBDE also satisfy our symmetric LBT. Moreover we have proved that there exists a solution which satisfies our trilinear equation but does not satisfy any of bilinear Hirota equations extended to the four-dimensional lattice.

An integrable trilinear partial differential equation (PDE) was discussed in [14], which has a connection with the Broer–Kaup system. Generalization to multilinear PDEs was also studied in [15] and the Painlevé properties were examined. The authors claim that no bilinear form corresponding to them has been found. Since the equations considered in these references are not difference but differential equations, we need further investigation to see the connection of our work to theirs.

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